Chapter 3

Model Reduction Concepts and Substructuring Approaches for Linear Systems .

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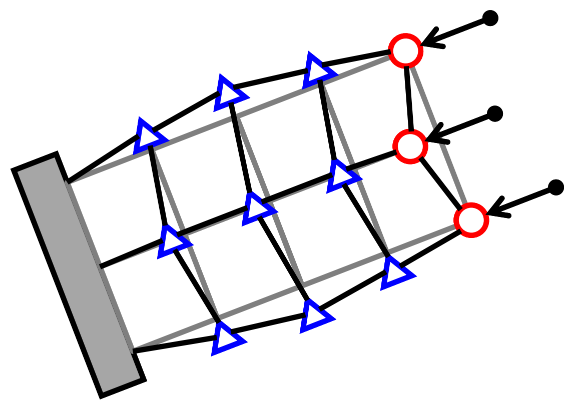


3.2.1 The Hurty-Craig-Bampton Method

Hurty, W.C.: Dynamic analysis of structural systems using component modes. AIAA journal **3**(4), 678–685 (1965)

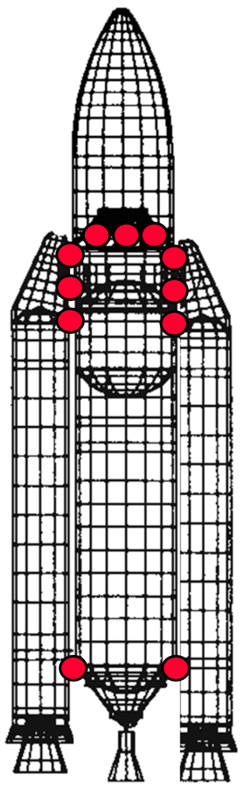
Bampton, M.C.C., Craig, J.R.R.: Coupling of substructures for dynamic analyses. AIAA Journal **6**(7), 1313–1319 (1968)

Dynamics of one substructure

$$\mathbf{M}^{(s)}\ddot{\mathbf{u}}^{(s)} + \mathbf{K}^{(s)}\mathbf{u}^{(s)} = \mathbf{f}^{(s)}$$

Partitioning in internal and boundary Dofs :

$$\begin{bmatrix} \mathbf{M}_{ii}^{(s)} & \mathbf{M}_{ib}^{(s)} \\ \mathbf{M}_{bi}^{(s)} & \mathbf{M}_{bb}^{(s)} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}}_i^{(s)} \\ \ddot{\mathbf{u}}_b^{(s)} \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{ii}^{(s)} & \mathbf{K}_{ib}^{(s)} \\ \mathbf{K}_{bi}^{(s)} & \mathbf{K}_{bb}^{(s)} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_i^{(s)} \\ \mathbf{u}_b^{(s)} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{f}_b^{(s)} \end{Bmatrix}$$

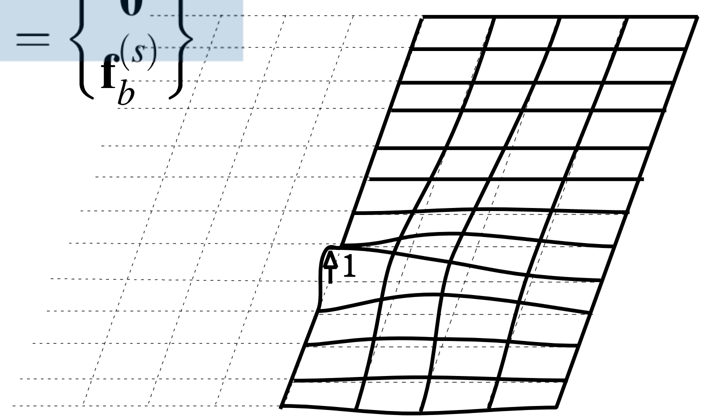


Representation modes (component modes) ?

First idea : static modes (Guyan-Irons) related to interface boundary Dofs

$$\begin{bmatrix} \mathbf{M}_{ii}^{(s)} & \mathbf{M}_{ib}^{(s)} \\ \mathbf{M}_{bi}^{(s)} & \mathbf{M}_{bb}^{(s)} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}}_i^{(s)} \\ \ddot{\mathbf{u}}_b^{(s)} \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{ii}^{(s)} & \mathbf{K}_{ib}^{(s)} \\ \mathbf{K}_{bi}^{(s)} & \mathbf{K}_{bb}^{(s)} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_i^{(s)} \\ \mathbf{u}_b^{(s)} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{f}_b^{(s)} \end{Bmatrix}$$

$$\boldsymbol{\Psi}^{(s)} = \begin{bmatrix} -\mathbf{K}_{ii}^{(s)^{-1}} \mathbf{K}_{ib}^{(s)} \\ \mathbf{I} \end{bmatrix}$$



Static mode

Representation modes (component modes) ?

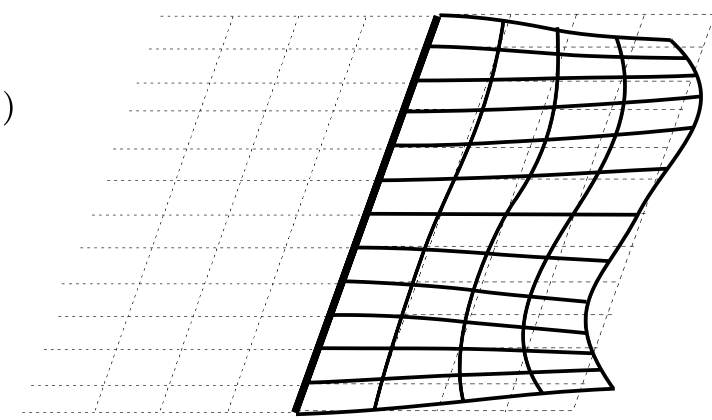
Improve static representation with fixed interface modes:

$$\begin{bmatrix} \mathbf{M}_{ii}^{(s)} & \mathbf{M}_{ib}^{(s)} \\ \mathbf{M}_{bi}^{(s)} & \mathbf{M}_{bb}^{(s)} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}}_i^{(s)} \\ \ddot{\mathbf{u}}_b^{(s)} \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{ii}^{(s)} & \mathbf{K}_{ib}^{(s)} \\ \mathbf{K}_{bi}^{(s)} & \mathbf{K}_{bb}^{(s)} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_i^{(s)} \\ \mathbf{u}_b^{(s)} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{f}_b^{(s)} \end{Bmatrix}$$

$$\boldsymbol{M}_{ii}^{(s)} \ddot{\boldsymbol{u}}_i^{(s)} + \boldsymbol{K}_{ii}^{(s)} \boldsymbol{u}_i^{(s)} = -\boldsymbol{M}_{ii}^{(s)} \ddot{\boldsymbol{u}}_b^{(s)} - \boldsymbol{K}_{ib}^{(s)} \boldsymbol{u}_b^{(s)}$$

$$\boldsymbol{u}_i^{(s)} = -\boldsymbol{K}_{ii}^{(s)^{-1}} \boldsymbol{K}_{ib}^{(s)} \boldsymbol{u}_b^{(s)} + \boldsymbol{u}_{i,dyn}^{(s)}$$

best represented by vibration modes of **fixed** interface

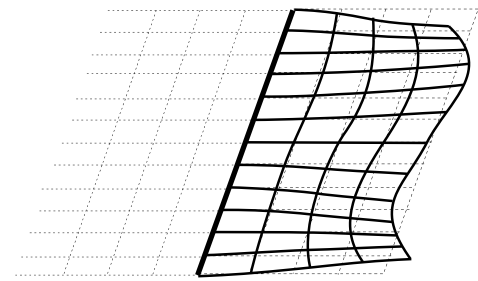


Vibration mode

Model Reduction / Numerical Techniques | Hurty-Craig-Bampton

$$\boldsymbol{u}_i^{(s)} = -\boldsymbol{K}_{ii}^{(s)^{-1}} \boldsymbol{K}_{ib}^{(s)} \boldsymbol{u}_b^{(s)} + \boldsymbol{u}_{i,dyn}^{(s)}$$

best represented by vibration modes of fixed interface



Vibration mode

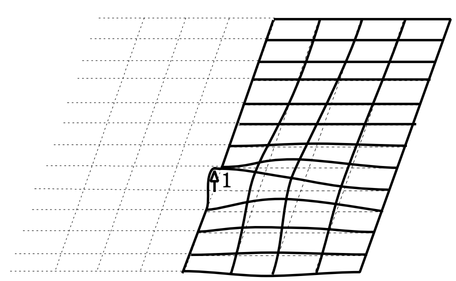
$$\left(\mathbf{K}_{ii}^{(s)} - \omega_r^2 \mathbf{M}_{ii}^{(s)}\right) \left\{\phi_i^{(s)}\right\}_r = 0$$

keeping only m modes

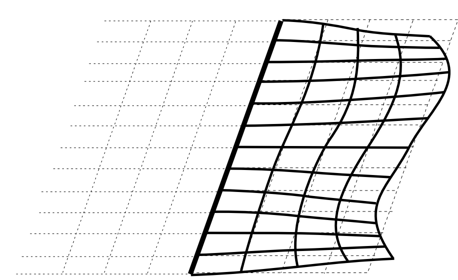
$$\boldsymbol{\Phi}^{(s)} = \begin{bmatrix} \left\{\phi_i^{(s)}\right\}_1, \ldots, \left\{\phi_i^{(s)}\right\}_m \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_i^{(s)} \\ \mathbf{0} \end{bmatrix}.$$

$$\begin{Bmatrix} \mathbf{u}_i^{(s)} \\ \mathbf{u}_b^{(s)} \end{Bmatrix} \approx \mathbf{T}^{\mathrm{HCB}(s)} \begin{Bmatrix} \mathbf{q}_i^{(s)} \\ \mathbf{u}_b^{(s)} \end{Bmatrix}$$

$$\mathbf{T}^{(\mathrm{s})\mathrm{HCB}} = \begin{bmatrix} \boldsymbol{\Phi}^{(s)} & \boldsymbol{\Psi}^{(s)} \end{bmatrix}$$



Static mode



Vibration mode

$$\mathbf{T}^{(s)\mathrm{HCB}} = \begin{bmatrix} \boldsymbol{\Phi}^{(s)} & \boldsymbol{\Psi}^{(s)} \end{bmatrix}$$

Reduced matrices for a substrucutre :

$$\mathbf{K}^{(s)\mathrm{HCB}} = \left(\mathbf{T}^{(s)\mathrm{HCB}}\right)^{\mathrm{T}} \mathbf{K}^{(s)} \mathbf{T}^{(s)\mathrm{HCB}} \qquad \mathbf{M}^{(s)\mathrm{HCB}} = \left(\mathbf{T}^{(s)\mathrm{HCB}}\right)^{T} \mathbf{M}^{(s)} \mathbf{T}^{(s)\mathrm{HCB}}$$

$$\boxed{\mathbf{K}^{(s)HCB} = \begin{bmatrix} \boldsymbol{\Omega}_m^{(s)^2} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{K}}_{bb}^{(s)} \end{bmatrix} \quad \text{and} \quad \mathbf{M}^{(s)HBC} = \begin{bmatrix} \mathbf{I} & \tilde{\mathbf{M}}_{ib}^{(s)} \\ \tilde{\mathbf{M}}_{bi}^{(s)} & \tilde{\mathbf{M}}_{bb}^{(s)} \end{bmatrix}}$$

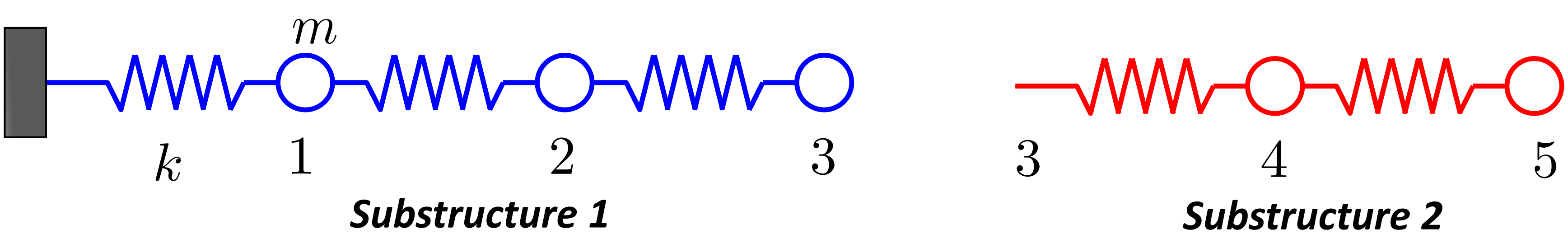
where

$$\tilde{\mathbf{K}}_{bb}^{(s)} = \mathbf{K}_{bb}^{(s)} - \mathbf{K}_{bi}^{(s)} \mathbf{K}_{bb}^{(s)^{-1}} \mathbf{K}_{bi}^{(s)}$$

$$\tilde{\mathbf{M}}_{bb}^{(s)} = \mathbf{M}_{bb}^{(s)} - \mathbf{M}_{bi}^{(s)} \mathbf{K}_{ii}^{(s)^{-1}} \mathbf{K}_{ib}^{(s)} - \mathbf{K}_{bi}^{(s)} \mathbf{K}_{ii}^{(s)^{-1}} \mathbf{M}_{ib}^{(s)} + \mathbf{K}_{bi}^{(s)} \mathbf{K}_{ii}^{(s)^{-1}} \mathbf{M}_{ii}^{(s)} \mathbf{K}_{ii}^{(s)^{-1}} \mathbf{K}_{ib}^{(s)}$$

$$\tilde{\mathbf{M}}_{ib} = \boldsymbol{\Phi}^{(s)^T} \left(\mathbf{M}_{ib}^{(s)} - \mathbf{M}_{ii}^{(s)} \mathbf{K}_{ii}^{(s)^{-1}} \mathbf{K}_{ib}^{(s)} \right) = \tilde{\mathbf{M}}_{bi}^T$$

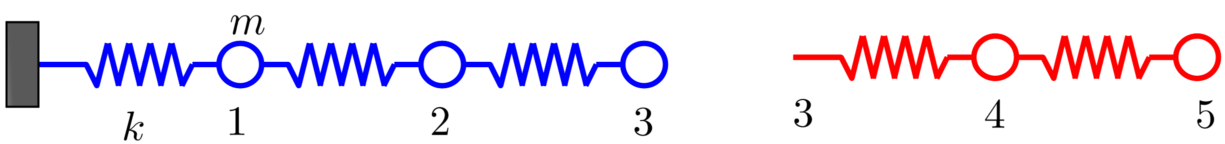
Let us consider the previous example:



$$\mathbf{M}^{(1)} = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \qquad \mathbf{M}^{(2)} = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{array}\right]$$

$$\mathbf{K}^{(1)} = \left[\begin{array}{r|rr} 1 & -1 & 0 \\ \hline -1 & 2 & -1 \\ 0 & -1 & 2 \end{array}\right] \qquad \mathbf{K}^{(2)} = \left[\begin{array}{r|rr} 1 & -1 & 0 \\ \hline -1 & 2 & -1 \\ 0 & -1 & 1 \end{array}\right]$$

N.B.: the mass at node 3 has already been included in substructure (1)



For the first substructure, the internal modes (fixing dof 3) and the Schur complement are, respectively:

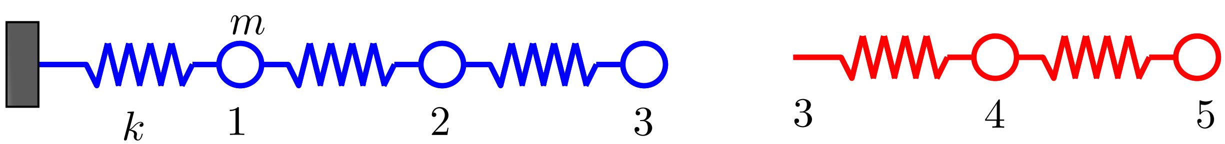
$$\mathbf{\Phi}^{(1)} = \begin{bmatrix} 0.7071 & -0.7071 \\ 0.7071 & 0.7071 \end{bmatrix} \qquad \mathbf{S}^{(1)} = \begin{bmatrix} 0.6667 \\ 0.3333 \end{bmatrix}$$

And the reduction matrix becomes:

$$\mathbf{R}^{(1)} = \begin{bmatrix} 1.0000 & 0 & 0 \\ 0.6667 & 0.7071 & -0.7071 \\ 0.3333 & 0.7071 & 0.7071 \end{bmatrix}$$

To give reduced mass and stiffness matrices:

$$\tilde{\mathbf{K}}^{(1)} = \left[\begin{array}{r|rr} 0.3333 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 3 \end{array}\right] \qquad \tilde{\mathbf{M}}^{(1)} = \left[\begin{array}{r|rr} 1.5556 & 0.7071 & -0.2357 \\ \hline 0.7071 & 1 & 0 \\ -0.2357 & 0 & 1 \end{array}\right]$$



For the second substructure, the internal modes (fixing dof 3) and the Schur complement are, respectively:

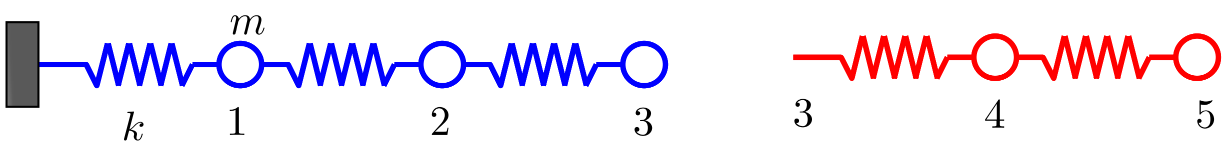
$$\mathbf{\Phi}^{(2)} = \begin{bmatrix} 0.7071 & -0.7071 \\ 1 & 1 \end{bmatrix} \qquad \mathbf{S}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

And the reduction matrix becomes:

$$\mathbf{R}^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0.7071 & -0.7071 \\ 1 & 1 & 1 \end{bmatrix}$$

To give reduced mass and stiffness matrices:

$$\tilde{\mathbf{K}}^{(2)} = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0.5858 & 0 \\ 0 & 0 & 3.4142 \end{array}\right] \qquad \tilde{\mathbf{M}}^{(2)} = \left[\begin{array}{c|cc} 1.5000 & 1.2071 & -0.2071 \\ \hline 1.2071 & 1 & 0 \\ -0.2071 & 0 & 1 \end{array}\right]$$



	Exact	**Guyan (static)**	**Craig Bampton (1 mode)**	**Craig Bampton (2 modes)**
Mode 1	0.3129	0.3303	0.3129	0.3129
Mode 2	0.9080	-	0.9080	0.9080
Mode 3	1.4142	-	1.4883	1.4142
Mode 4	1.7820	-	-	1.7821
Mode 5	1.9754	-	-	1.9753